

MULTISCALE ANALYSIS FOR A VECTOR-BORNE EPIDEMIC MODEL

MAX O. SOUZA

ABSTRACT. We investigate a classical mass-action model for diseases transmitted by arbovirus. Most traditional studies have focused on global stability issues and oscillatory behaviour. Here, we take a different view and we study the dynamics of the model, under the hypothesis that the typical timescale for the host and the vector are distinct. In this way, two asymptotic dynamics naturally appear: the fast host dynamics and the fast vector dynamics. The former yields, at leading order, a SIR model for the hosts, but with a modified incidence rate. Thus, the vector disappears from the model, and the dynamics is similar to a directly contagious disease. The latter yields a SI model for the vectors, with the hosts disappearing from the model. Numerical results show the performance of the approximation, and a rigorous proof validates the reduced models.

1. INTRODUCTION

1.1. Background. Vector-borne diseases are different from direct contagious ones since there is indirect transmission from host to the vector and vice-versa. From the point of view of mass-action epidemiological modelling, perhaps the most natural and simplest model is the coupling of SIR model for the host and SI model for the vector that was first developed by [1, 3]. It is given schematically in Figure 1

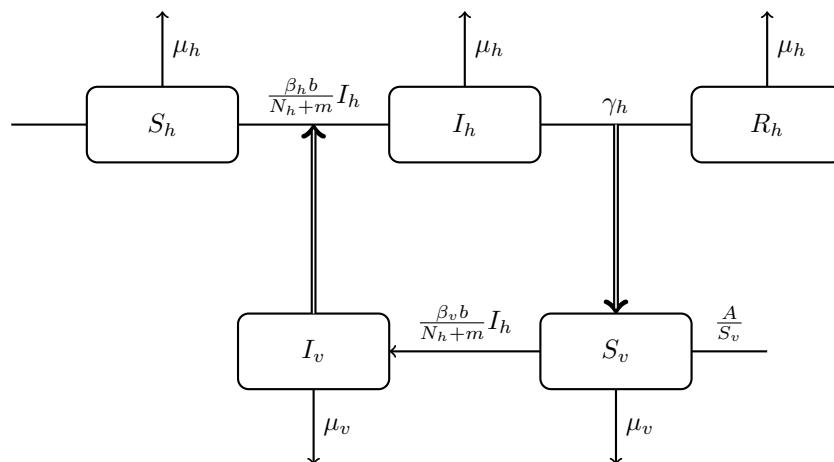


FIGURE 1. Compartmental description of the arbovirus model by [1, 3].

Date: August 10, 2011.

2000 Mathematics Subject Classification. Primary 92D30; Secondary 34E13.

Key words and phrases. Arboviruses, Dengue, Multiscale Asymptotics.

The author acknowledges many useful discussions held in the Dengue Modeling initiative developed at the CMA/FGV, where part of this work was performed. The author also acknowledges the workshops and support of PRONEX Dengue under CNPQ grant # 550030/2010-7. The author is partially supported by CNPq grant # 309616/2009-3 and FAPERJ grant # 110.174/2009.

Parameter	Meaning
N_h^* and N_v^*	Number of hosts and vectors
ν_h^* and ν_v^*	birth rate for hosts and vectors
β_h^* and β_v^*	probability of a host being infected by a vector and vice-versa.
γ^*	recovering rate
m^*	Number of alternative blood sources

TABLE 1. Description of parameters meaning in the compartmental model depicted in Figure 1.

The most straightforward translation from the compartmental model shown in Figure 1 is given by the following system:

$$(1) \quad \begin{cases} \dot{S}_h^* &= \mu_h^*(N_h^* - S_h^*) - \frac{\beta_h^* b^*}{N_h^* + m^*} S_h^* I_v^* \\ \dot{I}_h^* &= \frac{\beta_h^* b^*}{N_h^* + m^*} S_h^* I_v^* - (\mu_h^* + \gamma^*) I_h^* \\ \dot{R}_h^* &= \gamma^* I_h^* - \mu_h^* R_h^* \\ \dot{S}_v^* &= A^* - \mu_v^* S_v^* - \frac{\beta_v^* b^*}{N_h^* + m^*} S_v^* I_h^* \\ \dot{I}_v^* &= \frac{\beta_v^* b^*}{N_h^* + m^*} S_v^* I_h^* - \mu_v^* I_v^* \end{cases}$$

Here starred variables indicate dimensional quantities. The variables S , I and R have the usual epidemiological meaning, with the subscript indicating if they refer to the hosts or to the vectors. For the other parameters, we refer to Table 1.

System (1) has been comprehensively studied by [4], and extensively used for studies of Dengue as described, for instance, in [5]. Most theoretical studies have focused in the global stability features of System (1). In particular a proof of global stability using the theory of competitive systems can be found in [4]. More recently, some partial Lyapunov proofs were also available in [2] and [7]. Also a proof using a Lyapunov approach for the disease free equilibrium, and a persistence argument for the endemic equilibrium can be found in [8].

In this study, we shall take a different route and first show that there are two natural and independent timescales in System 1, namely the host timescale for the disease evolution and the correspondingly one for the vector. We then examine the possibility that these two scales are asymptotically separated and find that, for fast vector dynamics, System (1) can be approximated by a 2-D SIR model, with rational incidence rates. For fast host dynamics, the simplification is even more dramatic, and System (1) can be approximated by a 1-D SI model, again with a rational incidence rate. The derived approximations are uniform for all positive times. Incidentally, some results on global stability are also presented.

1.2. Outline. In Section 2, we describe the basic model studied and review some of its properties. A global analysis using only a Lyapunov approach is also presented. In Section 3, we study the *Fast Vector Dynamics* limit, i.e.,

$$\sigma, \mu_v \sim \epsilon^{-1},$$

with all the remaining nondimensional parameters $\mathcal{O}(1)$. This yields a reduced SIR system with modified nonlinear incidence rates. The dual limit, the *Fast Host Dynamics*, i.e.

$$\delta, \mu_h, \gamma \sim \epsilon^{-1},$$

with all the remaining nondimensional parameters $\mathcal{O}(1)$ yields a reduced SI system that is somewhat less interesting. The latter limit is presented briefly in Appendix A. Conclusions are presented in Section 4.

2. PRELIMINARIES

2.1. Scalings and adimensionalisation. We adimensionalise System 1 by letting

$$(S_h^*, I_h^*, R_h^*) = N_h^* (S_h, I_h, R_h) \quad \text{and} \quad (S_v^*, I_v^*) = \frac{A^*}{\mu_v^*} (S_v, I_v).$$

Also

$$t^* = (\Delta^*)^{-1} t,$$

where Δ^* is, for the moment, an arbitrary time-scale. We immediately obtain the new System

$$\begin{aligned}\dot{S}_h &= \mu_h(1 - S_h) - \delta S_h I_v \\ \dot{I}_h &= \delta S_h I_v - (\mu_h + \gamma) I_h \\ \dot{R}_h &= \gamma I_h - \mu_h R_h \\ \dot{S}_v &= \mu_v(1 - S_v) - \sigma S_v I_h \\ \dot{I}_v &= \sigma S_v I_h - \mu_v I_v\end{aligned}$$

where

$$\delta = \frac{\beta_h^* b^* A^*}{\mu_v^* \Delta^* (N_h^* + m^*)} \quad \text{and} \quad \sigma = \frac{\beta_v^* b^* N_h^*}{\Delta^* (N_h^* + m^*)}$$

Also

$$\gamma = \frac{\gamma^*}{\Delta^*}, \quad \mu_h = \frac{\mu_h^*}{\Delta^*} \quad \text{and} \quad \mu_v = \frac{\mu_v^*}{\Delta^*}.$$

Typically, Δ^* will be chosen such that at least one of γ , μ_h , μ_v are order one. We shall not make any a priori choice, but indicate appropriate choices when studying particular regimes.

As showed in [4], the dynamics is restricted to the nonnegative orthant, and it is conservative:

Lemma 1. *Let $\mathbb{R}_{\geq 0}^5$ denote the nonnegative orthant of \mathbb{R}^5 . Then a solution of System (1), with an initial condition in $\mathbb{R}_{\geq 0}^5$, stays in $\mathbb{R}_{\geq 0}^5$. Moreover, the sums $S_h^* + I_h^* + R_h^*$ and $S_v^* + I_v^*$ are preserved by the evolution.*

Because of Lemma 1, if the initial conditions for the host fractions add to one, with the same being true for the vector fractions initial conditions, then this will be preserved by the evolution. Therefore, we can work with the simplified, but equivalent, model below:

$$(2) \quad \begin{cases} \dot{X} &= \mu_h(1 - X) - \delta X Z \\ \dot{Y} &= \delta X Z - (\mu_h + \gamma) Y \\ \dot{Z} &= \sigma(1 - Z) Y - \mu_v Z \end{cases}$$

2.2. Global stability analysis. System (2) has the following two equilibrium points

- (1) The disease free equilibrium: $X^* = 1$, $Y^* = Z^* = 0$.
- (2) The endemic equilibrium:

$$X^* = \frac{(\mu_h + \gamma)\mu_v + \mu_h\sigma}{(\mu_h + \delta)\sigma}, \quad Y^* = \frac{\mu_h[\delta\sigma - (\mu_h + \gamma)\mu_v]}{(\mu_h + \delta)(\mu_h + \gamma)\sigma}, \quad Z^* = \frac{\mu_h[\delta\sigma - (\mu_h + \gamma)\mu_v]}{\delta(\mu_h + \gamma)\mu_v + \delta\mu_h\sigma}$$

The endemic equilibrium can be conveniently written as

$$X^* = \frac{1}{R_0} \frac{1 + R_0 D_0}{1 + D_0}, \quad Y^* = \frac{\mu_h}{\mu_h + \gamma} \frac{R_0 - 1}{R_0 + R_0 D_0}, \quad Z^* = D_0 \frac{R_0 - 1}{1 + R_0 D_0}$$

where

$$R_0 = \frac{\sigma\delta}{\mu_v(\mu_h + \gamma)} \quad \text{and} \quad D_0 = \frac{\mu_h}{\delta}.$$

or as

$$X^* = \frac{1}{R_0} \frac{1 + R_0 D_0}{1 + D_0}, \quad Y^* = \frac{\mu_h}{\mu_h + \gamma} (1 - X^*), \quad Z^* = D_0 (1 - X^*)$$

Theorem 1. *Let R_0 be defined as above. Then for $R_0 \leq 1$ the disease-free equilibrium is globally asymptotic stable, while for $R_0 > 1$ the endemic equilibrium is globally asymptotic stable.*

Proof. Suppose $R_0 \leq 1$ and consider the following Lyapunov function

$$V(X, Y, Z) = X - \log X + Y + \frac{\delta}{\mu_v} Z.$$

Then

$$\dot{V} = \dot{X} \left(1 - \frac{1}{X}\right) + \dot{Y} + \dot{Z}$$

On substituting

$$\dot{V} = -\mu_h \frac{(1-X)^2}{X} - (\mu_h + \gamma)(1-R_0)Y - R_0ZY,$$

which is negative for $0 \leq R_0 \leq 1$, and X, Y, Z in the interior of the positive octant of \mathbb{R}^3 , other than $(X, Y, Z) = (1, 0, 0)$.

For $R_0 > 1$, let

$$V(X, Y, Z) = X - X^* \log \frac{X}{X^*} + Y - Y^* \log \frac{Y}{Y^*} + \frac{\delta X^*}{\mu_v} \left(Z - Z^* \log \frac{Z}{Z^*} \right).$$

Then we have

$$\begin{aligned} \dot{V} &= \dot{X} \left(1 - \frac{X^*}{X}\right) + \dot{Y} \left(1 - \frac{Y^*}{Y}\right) + \frac{\delta X^*}{\mu_v} \dot{Z} \left(1 - \frac{Z^*}{Z}\right) \\ &= \mu_h \left[1 + X^* - X - \frac{X^*}{X} \right] + (\mu_h + \gamma)Y^* + \delta X^*Z + \frac{\sigma \delta X^*}{\mu_v} (1-Z)Y + \\ &\quad - \delta \frac{XZY^*}{Y} - (\mu_h + \gamma)Y - \frac{\sigma \delta X^* Z^*}{\mu_v} (1-Z) \frac{Y}{Z} \\ &= \mu_h \left[3 - X^* - X - \frac{X^*}{X} \right] + (\mu_h + \gamma) (R_0 X^* (1 - Z^*) - 1) Y - \frac{\sigma \delta X^*}{\mu_v} ZY \\ &\quad - \frac{\sigma \delta X^* Z^*}{\mu_v} \frac{Y}{Z} - \frac{\delta XZY^*}{Y}. \end{aligned}$$

Since

$$R_0(1 - Z^*) = R_0 \frac{1 + D_0}{1 + R_0 D_0} = \frac{1}{X^*},$$

we are left with

$$\dot{V} = \mu_h \left[3 - X^* - X - \frac{X^*}{X} \right] - \frac{\sigma \delta X^*}{\mu_v} ZY - \frac{\sigma \delta X^* Z^*}{\mu_v} \frac{Y}{Z} - \frac{\delta XZY^*}{Y}.$$

Writing

$$X + \frac{X^*}{X} = X + \frac{(X^*)^2}{X} - \frac{X^*(1 - X^*)}{X},$$

and on using that

$$X + \frac{(X^*)^2}{X} \geq 2X^*,$$

we find

$$\dot{V} \leq 3\mu_h(1 - X^*) - \mu_h(1 - X^*) \frac{X^*}{X} - \frac{\sigma \delta X^*}{\mu_v} ZY - \frac{\sigma \delta X^* Z^*}{\mu_v} \frac{Y}{Z} - \frac{\delta XZY^*}{Y}.$$

This can be rewritten as

$$\dot{V} \leq 3\mu_h(1 - X^*) - \frac{\sigma \delta X^*}{\mu_v} YZ - \mu_h(1 - X^*) \frac{X^*}{X} - \frac{\mu_h \sigma (1 - X^*)}{\mu_v} \frac{Y}{Z} - \frac{\delta \mu_k (1 - X^*)}{\mu_h + \gamma} \frac{XZ}{Y}.$$

By the Arithmetic-Geometric inequality, we have

$$-\mu_h(1 - X^*) \frac{X^*}{X} - \frac{\mu_h \sigma (1 - X^*)}{\mu_v} \frac{Y}{Z} - \frac{\delta \mu_k (1 - X^*)}{\mu_h + \gamma} \frac{XZ}{Y} \leq -3\mu_h(1 - X^*) (R_0 X^*)^{1/3}.$$

Hence,

$$\dot{V} \leq 3\mu_h(1 - X^*) \left(1 - (R_0 X^*)^{1/3}\right) - \frac{\sigma \delta X^*}{\mu_v} YZ.$$

Since

$$R_0 X^* = \frac{1 + R_0 D_0}{1 + D_0} > 1, \quad \text{since } R_0 > 1,$$

we find that $\dot{V} < 0$ in the interior. \square

3. THE FAST VECTOR DYNAMICS

If we assume that the vector population typical times are much shorter than the typical times for the host, we might expect that the vector population will be nearly in equilibrium, and hence that $\dot{Z} = 0$. In this case, we formally obtain from (2)

$$(3) \quad \begin{cases} \dot{X} &= \mu_h(1 - X) - \delta \frac{\sigma XY}{\sigma Y + \mu_v} \\ \dot{Y} &= \delta \frac{\sigma XY}{\sigma Y + \mu_v} - (\mu_h + \gamma)Y \\ \dot{Z} &= 0 \end{cases}$$

Equation (3) can be recast as a SIR system with a modified incidence rate. While the derivation above is heuristic, we shall see that it can be obtained from a consistent multiscale asymptotic expansion. Moreover, the formal expansion can be rigorously justified. In what follows, we shall write

$$\sigma = \bar{\sigma}\epsilon^{-1} \quad \text{and} \quad \mu_v = \bar{\mu}_v\epsilon^{-1}.$$

Then, we are interested in solve

$$(4) \quad \begin{cases} \dot{X} &= \mu_h(1 - X) - \delta XZ \\ \dot{Y} &= \delta XZ - (\mu_h + \gamma)Y \\ \epsilon \dot{Z} &= \bar{\sigma}(1 - Z)Y - \bar{\mu}_v Z \end{cases}$$

subject to the initial condition

$$X(0) = X_0, \quad Y(0) = y_0 \quad \text{and} \quad Z(0) = Z_0.$$

3.1. Asymptotic expansion. Let

$$\epsilon\tau = t.$$

Then, we seek an expansion of the form

$$\begin{aligned} X &= X^0(t) + \mathcal{O}(\epsilon) \\ Y &= Y^0(t) + \mathcal{O}(\epsilon) \quad \text{and} \\ Z &= Z^0(t) + \hat{Z}^0(\tau) + \mathcal{O}(\epsilon), \end{aligned}$$

where

$$\lim_{\tau \rightarrow \infty} \hat{Z}^0(\tau) = 0.$$

The leading order equations are given by the following differential-algebraic system:

$$\begin{aligned} X_t^0 &= \mu_h(1 - X^0) - \delta X^0 Z^0 \\ Y_t^0 &= \delta X^0 Z^0 - (\mu_h + \gamma)Y^0 \\ 0 &= \bar{\sigma}(1 - Z^0)Y^0 - \bar{\mu}_v Z^0. \end{aligned}$$

This yields

$$Z^0(t) = \frac{\bar{\sigma}Y^0(t)}{\bar{\sigma}Y^0(t) + \bar{\mu}_v}.$$

and hence, we obtain the system:

$$(5) \quad \begin{cases} X_t^0 &= \mu_h(1 - X^0) - \delta \frac{\bar{\sigma}X^0Y^0}{\bar{\sigma}Y^0 + \bar{\mu}_v} \\ Y_t^0 &= \delta \frac{\bar{\sigma}X^0Y^0}{\bar{\sigma}Y^0 + \bar{\mu}_v} - (\mu_h + \gamma)Y^0 \end{cases}$$

with initial condition $X^0(0) = X_0$ and $Y^0(0) = Y_0$. To solve for \hat{Z}^0 , we notice that

$$X^0(t) = X^0(\epsilon\tau) = X^0(0) + \epsilon\tau X_t^0(0) + \mathcal{O}(\epsilon^2),$$

with similar expansions for $Y^0(t)$ and $Z^0(t)$.

Therefore, we find that \hat{Z}^0 satisfies

$$\hat{Z}_\tau^0 = -(\bar{\sigma}Y_0 + \bar{\mu}_v)\hat{Z}^0,$$

i.e

$$\hat{Z}^0(\tau) = \left(Z_0 - \frac{\bar{\sigma}Y_0}{\bar{\sigma}Y_0 + \bar{\mu}_v} \right) e^{-(\bar{\sigma}Y_0 + \bar{\mu}_v)\tau}.$$

3.2. Global stability analysis of the asymptotic system. First, we have the following basic result

Proposition 1. *Let*

$$\mathcal{S} = \{x + y \leq 1, \quad x, y \geq 0\}.$$

Then \mathcal{S} is invariant by the flow of (5). In particular, the corresponding solutions are global in time.

Proof. Since $Y^0 = 0$ is an invariant set, solution with $y \geq 0$ will remain so. Also, when $X^0 = 0$, the flow points inside \mathcal{S} . Thus $x \geq 0$. Finally, $(X^0 + Y^0)_t \leq 0$. \square

System (5) has two equilibrium points in the positive quadrant:

- (1) $X_* = 1, Y_* = 0$.
- (2)

$$X_* = \frac{(\nu_h + \gamma)\bar{\mu}_v + \nu_h\bar{\sigma}}{(\mu_h + \delta)\bar{\sigma}} \quad \text{and} \quad Y_* = \frac{\delta\mu_n\bar{\sigma} - (\mu_h^2 + \gamma\mu_h)\bar{\mu}_v}{\mu_h^2 + (\gamma + \delta)h + \delta\gamma}\bar{\sigma}.$$

Remark 1. *Notice that $X_* = X^*$ and $Y_* = Y^*$. Thus the equilibrium of (5) correspond to the projections in the XY plane of the equilibria of (2). Notice also that*

$$Z_* = \frac{\bar{\sigma}Y_*}{\bar{\sigma}Y_* + \bar{\mu}_v} = Z^*.$$

We then have

Theorem 2. *Let*

$$\bar{R}_0 = \frac{\delta\bar{\sigma}}{\bar{\mu}_v(\nu_h + \gamma)}$$

Then $\bar{R}_0 = R_0$, and for $R_0 \leq 1$ the disease free equilibrium is globally asymptotically stable. For $R_0 > 1$, the endemic equilibrium is globally asymptotically stable.

Proof. Let

$$F(Y_0) = \frac{\bar{\sigma}Y_0 + \bar{\mu}_v}{Y_0}$$

Then F is a Dulac function for system (5) in proper compact subset of \mathcal{S} , since

$$\partial_{X_0}[\mu_h(1 - X_0)F(Y_0) - \delta X_0] + \partial_{Y_0}[\delta X_0 - (\mu_h + \gamma)(\bar{\sigma}Y_0 + \bar{\mu}_v)] = -\mu_h F(Y_0) - \delta - \bar{\sigma}(\mu_h + \gamma) < 0.$$

Thus, the system cannot have a closed orbit in the interior of \mathcal{S} . For $R_0 \leq 1$, the only equilibrium in \mathcal{S} is $(1, 0)$. Thus by all orbits must converge to this equilibrium point.

The linearisation of (5) is

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}_T = \begin{pmatrix} -\mu_h - \delta \frac{\bar{\sigma}Y_0}{\bar{\sigma}Y_0 + \bar{\mu}_v} & -\bar{\sigma}\delta X_0 \frac{\bar{\mu}_v}{(\bar{\sigma}Y_0 + \bar{\mu}_v)^2} \\ \delta \frac{\bar{\sigma}Y_0}{\bar{\sigma}Y_0 + \bar{\mu}_v} & \bar{\sigma}\delta X_0 \frac{\bar{\mu}_v}{(\bar{\sigma}Y_0 + \bar{\mu}_v)^2} - \mu_h - \gamma \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$$

For the disease free equilibrium, the eigenvalues of Jacobian are $-\mu_h$ and $(\mu_h + \gamma)(R_0 - 1)$. Thus, the disease free equilibrium is a locally asymptotically stable node for $R_0 < 1$, and a saddle for $R_0 > 1$. The unique orbit that approaches the disease free it easily shown to be the intersection of \mathcal{S} with $Y_0 = 0$. Thus, all the other orbits must approach the endemic equilibrium. \square

3.3. Asymptotic convergence. The asymptotic expansions derived in 3.1 can be shown to be indeed asymptotic. The ideas used here are similar to the ones used to formalise Kinetic Menton's theory—cf. [6] and references therein for instance. In particular, we have the following:

Theorem 3. *Let $W(t) = (X(t), Y(t), Z(t))$ and $W^0(t, \tau) = (X^0(t), Y^0(t), Z^0(t) + \hat{Z}^0(\tau))$. Denote the uniform norm in $[0, \infty)$ by $\|\cdot\|_\infty$. Then, for sufficient small $\epsilon > 0$, there exists a constant $C > 0$, independent of ϵ , such that*

$$\|W - W^0\|_\infty \leq C\epsilon.$$

The proof of Theorem 3 is divided in several lemmas.

We write system (5) in a more concise form as

$$\begin{aligned}\dot{\mathbf{W}} &= \mathcal{F}(\mathbf{W}, Z), \\ \dot{Z} &= \mathcal{G}(\mathbf{W}, Z).\end{aligned}$$

with $\mathbf{W}(t) = (X(t), Y(t))^t$ and \mathcal{F} and \mathcal{G} being the appropriate entries of the right hand side of (4). We shall write

$$\mathbf{W} = \mathbf{W}^0 + \epsilon \hat{\mathbf{W}}^0 + \epsilon \mathbf{Q} \quad \text{and} \quad Z = Z^0 + \hat{Z}^0 + \epsilon \bar{Z},$$

with

$$\hat{\mathbf{W}}^0 = \frac{X_0}{\bar{\sigma}Y_0 + \bar{\mu}_v} \left(Z_0 - \frac{\bar{\sigma}Y_0}{\bar{\sigma}Y_0 + \bar{\mu}_v} \right) e^{-(\bar{\sigma}Y_0 + \bar{\mu}_v)t/\epsilon} (1, -1)^t.$$

Notice that since $\hat{\mathbf{W}}^0$ is bounded, we need only to prove that (\mathbf{Q}, \bar{Z}) exist, are bounded, and are unique. In this case, we can then take $C = \|(\mathbf{Q}, \bar{Z})\|_\infty$. First, we observe that

$$\dot{\mathbf{W}}^0 + \dot{\hat{\mathbf{W}}}^0 = \mathcal{F}(\mathbf{W}^0, Z^0 + \hat{Z}^0) + K(t) \quad \text{and} \quad \dot{Z}^0 + \dot{\hat{Z}}^0 = \mathcal{G}(\mathbf{W}^0, Z^0 + \hat{Z}^0) + L(t),$$

where

$$K(t) = \delta \hat{Z}^0(t/\epsilon)(X^0(t) - X_0) \quad \text{and} \quad L(t) = \bar{\sigma} \hat{Z}^0(t/\epsilon)(Y^0(t) - Y_0).$$

In particular, because of the fast decay of \hat{Z}^0 , and because $L(0) = K(0) = 0$, it follows that there exists a constant $C > 0$, such that

$$\int_0^\infty K(t) dt, \quad \int_0^\infty L(t) dt \leq C\epsilon.$$

Since \mathcal{F} and \mathcal{G} are quadratic, we write:

$$\begin{aligned}\mathcal{F}(\mathbf{W}, Z) &= \mathcal{F}(\mathbf{W}^0, Z^0 + \hat{Z}^0) + \epsilon D_{\mathbf{W}}\mathcal{F}(\mathbf{W}^0, Z^0 + \hat{Z}^0)\mathbf{Q} + \epsilon D_Z\mathcal{F}(\mathbf{W}^0, Z^0 + \hat{Z}^0)\bar{Z} + \\ &\quad + \epsilon^2 \delta \bar{X} \bar{Y} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \\ \mathcal{G}(\mathbf{W}, Z) &= \mathcal{G}(\mathbf{W}^0, Z^0 + \hat{Z}^0) + \epsilon D_{\mathbf{W}}\mathcal{G}(\mathbf{W}^0, Z^0 + \hat{Z}^0)\mathbf{Q} + \epsilon D_Z\mathcal{G}(\mathbf{W}^0, Z^0 + \hat{Z}^0)\bar{Z} + \\ &\quad + \epsilon^2 \bar{\sigma} \bar{Y} Z.\end{aligned}$$

where $\mathbf{Q} = (\bar{X}, \bar{Y})^t$.

Let $T(t, s)$ be the fundamental solution to the linearised system

$$(6) \quad \begin{pmatrix} \dot{\mathbf{Q}} \\ \dot{\bar{Z}} \end{pmatrix} = \begin{pmatrix} D_{\mathbf{W}}\mathcal{F}(\mathbf{W}^0, Z^0 + \hat{Z}^0) & D_Z\mathcal{F}(\mathbf{W}^0, Z^0 + \hat{Z}^0) \\ D_{\mathbf{W}}\mathcal{G}(\mathbf{W}^0, Z^0 + \hat{Z}^0) & D_Z\mathcal{G}(\mathbf{W}^0, Z^0 + \hat{Z}^0) \end{pmatrix} \begin{pmatrix} \mathbf{Q} \\ \bar{Z} \end{pmatrix}$$

Then direct integration yields

Lemma 2. *The functions (\mathbf{Q}, \bar{Z}) satisfy the following integral equation:*

$$(7) \quad \begin{aligned} \begin{pmatrix} \mathbf{Q} \\ R \end{pmatrix} = & T(t, 0) \begin{pmatrix} Q_0 \\ R_0 \end{pmatrix} + \epsilon \delta \int_0^t T(t, s) \begin{pmatrix} \delta \bar{X} \bar{Y} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \bar{\sigma} \bar{Y} Z \end{pmatrix} ds + \\ & - \int_0^t T(t, s) \begin{pmatrix} \hat{\mathbf{W}}^0(s) \\ 0 \end{pmatrix} ds - \frac{1}{\epsilon} \int_0^t T(t, s) \begin{pmatrix} K(s) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ L(s) \end{pmatrix} ds. \end{aligned}$$

Moreover, the last term is bounded uniformly in ϵ .

We also have the following large time behaviour result for the linearised system (6):

Lemma 3. *Let $(\mathbf{Q}(t), \bar{Z}(t))$ a solution to (6). Then*

$$\lim_{t \rightarrow \infty} (\mathbf{Q}(t), \bar{Z}(t)) = \mathbf{0}.$$

In particular, the solutions to (6) are bounded uniformly in time any given ϵ . Moreover, they are also uniformly bounded in $\epsilon \leq 1$, for all $t \geq 0$.

Proof. For notation convenience, let us write (6) as

$$\begin{pmatrix} \dot{\mathbf{Q}} \\ \dot{\bar{Z}} \end{pmatrix} = A(\mathbf{W}^0, Z^0 + \hat{Z}^0) \begin{pmatrix} \mathbf{Q} \\ \bar{Z} \end{pmatrix}.$$

Fix $\epsilon > 0$ and (X_0, Y_0, Z_0) . From Theorem 2, we know that

$$\lim_{t \rightarrow \infty} (\mathbf{W}^0(t), Z^0(t) + \hat{Z}^0(t/\epsilon)) = (\mathbf{W}^*, Z^*),$$

where (\mathbf{W}^*, Z^*) will be the globally asymptotic stable equilibrium given by Theorem 1. Therefore, there exists $T > 0$, such that $t > T$ implies that A is negative-definite. Since $\hat{Z}^0 \rightarrow 0$, as $\epsilon \rightarrow 0$. We can choose T such this holds for all $0 < \epsilon \leq 1$.

Because of the continuity of the solution with respect to the initial conditions, we have that T is a continuous function of the initial conditions. Since these lie on a compact set, we can pick T such that A is negative definite for all $0 < \epsilon \leq 1$ and for all (X_0, Y_0, Z_0) , such that $X_0, Y_0 \geq 0$, $X_0 + Y_0 \leq 1$ and $0 < Z_0 \leq 1$. But then, for any such (X_0, Y_0, Z_0) and $0 < \epsilon \leq 1$, we have that

$$\lim_{t \rightarrow \infty} T(t, T) = 0.$$

Therefore, for any initial condition $(Q_0, R_0)^t$, we shall have

$$\lim_{t \rightarrow \infty} T(t, 0) \begin{pmatrix} Q_0 \\ R_0 \end{pmatrix} = \lim_{t \rightarrow \infty} T(t, T) T(T, 0) \begin{pmatrix} Q_0 \\ R_0 \end{pmatrix} = 0.$$

□

Proof of Theorem 3. First, we observe that the nonlinear term in (7) is locally Lipschitz, hence a standard fixed point yields existence and uniqueness for $0 \leq t < t_0$, for some, possibly small, t_0 .

From the decaying of $\hat{\mathbf{W}}^0$ and from Lemma 2, we conclude that the two last terms of (7) are uniformly bounded in time and ϵ .

Moreover, Lemma 3 implies that the first two terms on the right hand side of (7) are also uniformly bounded in time and in ϵ , for sufficiently small ϵ . Thus, we conclude that the same also holds true for $(Q, R)^t$. Therefore, the solution to (7) is globally defined in time, and it is bounded uniformly in ϵ , if the latter is sufficiently small.

□

3.4. Numerical results. We now present some numerical results in Figure 2. There we compare the full model with the asymptotic model. As expected, the approximation of $X(t)$ by $X^0(t)$ and of $Y(t)$ by $Y^0(t)$ are indeed uniform for all time, while the approximation of $Z(t)$ by $Z^0(t)$ fail to be uniform in an initial layer. Notice also that such nonuniform behaviour is suppressed by including the corrector term in the initial layer.

Remark 2. *We observe that the structure of the asymptotic solution shows that the solutions must be confined, except for an initial layer, on what is usually termed a slow manifold. This behaviour is indicated in Figure 4.*

4. CONCLUDING REMARKS

Diseases that are vector-borne have a number of features that distinguish them of contagious ones. Typically, timescales for the dynamics of the host and vectors are not within the same order, since mosquitos, for instance, that are a prevalent vector for such diseases have a very fast life cycle compared to humans say. With this in mind, we investigated the dynamical consequences of having host and vector dynamics with distinct timescales in the classical arbovirus model introduced by [1, 3]. The natural regimes to look in this model are the *fast vector dynamics* (FVD) and *fast host dynamics* (FHD). While the former seems to be the most natural choice, we take the view that there might be scenarios where the latter may be observed.

By means of a formal multiscale asymptotic analysis, we study both regimes. For the FVD, we find the leading order dynamics yields a SIR model for the host, with a modified incidence rate. Thus, the vector is removed from the model being present only parametrically as a function of the host infected fraction. Such a relationship, appart from its mathematical interest, might also be useful in verifying if field data is conformant, within the model, with the regime hypothesis. Additionally, the FHD regime yields an even more dramatic reduction with an SI model for the vectors, again with a modified incidence rate. Numerical results presented show that the approximation is indeed uniformly asymptotic in time.

An interesting feature of the studied regimes is that they do not imply any condition on R_0 , and hence are compatible with a variety of disease developments from the point of view of global dynamics. Indeed, for both reduced models, the equilibria are preserved by the asymptotic approximation, and the global stability dynamics is consistent with the global stability dynamics of the full model. Finally, we have confirmed rigorously the asymptotic character of the approximation up to the derived order.

The results presented here suggest that other epidemiological models of vector-borne diseases might be studied in a similar fashion. Hopefully, this procedure would provide model reductions allowing a better understanding and, perhaps, even a classification scheme for different arboviruses diseases based on the relative timescales of infection, the host and the vector dynamics.

APPENDIX A. THE FAST HOST DYNAMICS

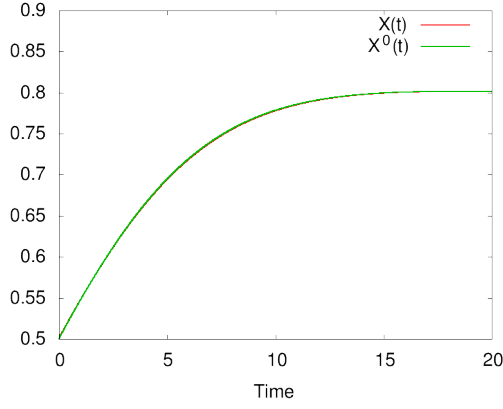
Analogous to the previous case, we now assume that the dynamics of the host is much faster than the dynamics of the vector. In this case, one might expect that host populations is nearly in equilibrium. Hence, we should have $\dot{X} \approx 0$ and $\dot{Y} \approx 0$, i.e, we should have the system

$$(8) \quad \begin{cases} 0 &= \mu_h(1 - X) - \delta \frac{\sigma XY}{\sigma Y + \mu_v} \\ 0 &= \delta \frac{\sigma XY}{\sigma Y + \mu_v} - (\mu_h + \gamma)Y \\ \dot{Z} &= \sigma(1 - Z)Y - \mu_v Z \end{cases}$$

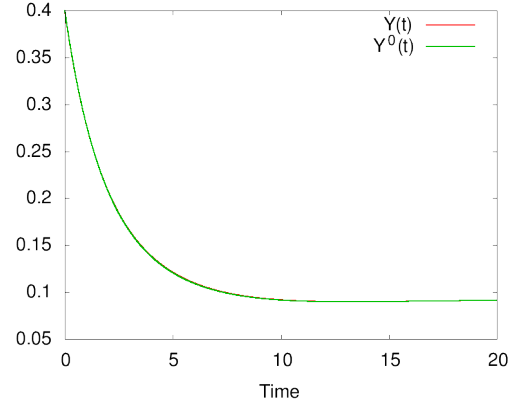
Equation (8) can be seen as SI system for the vector, with a modified incidence rate.

In order to justify System (8), we assume that

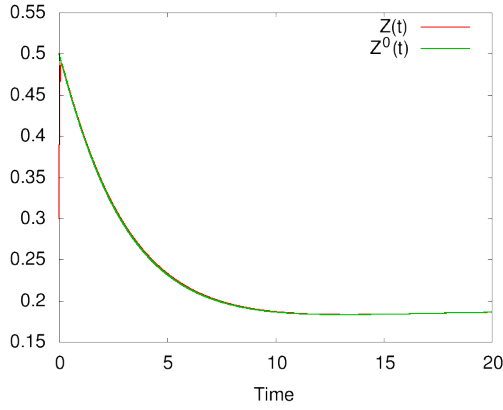
$$\delta = \bar{\delta}\epsilon^{-1}, \quad \mu_h = \bar{\mu}_h\epsilon^{-1} \quad \text{and} \quad \gamma = \bar{\gamma}\epsilon^{-1}.$$



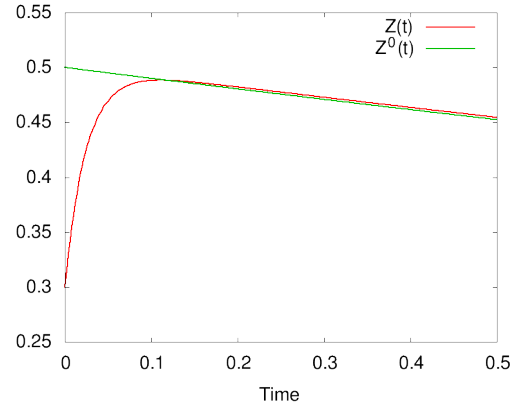
(A) Susceptible hosts



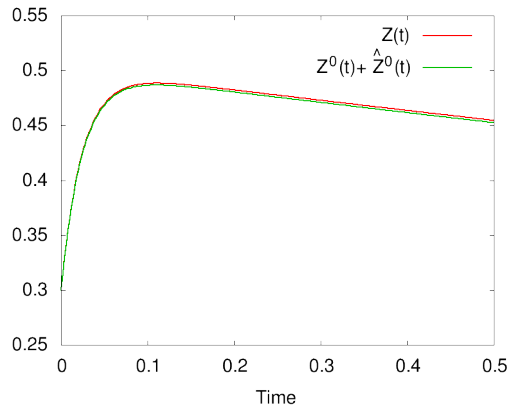
(B) Infected hosts



(C) Infected vectors—leading order approximation



(D) Infected vectors—blow-up at origin



(E) Infected vectors—composite approximation

FIGURE 2. Comparison between the full and asymptotic models. Here $R_0 \approx 1.58$, $D_0 = 0.75$, $\epsilon = 0.01$. Also, $\mu_h = 0.3$, $\delta = 0.4$, $\gamma = 0.35$, $\bar{\sigma} = 0.5$ and $\bar{\mu}_v = 0.2$.

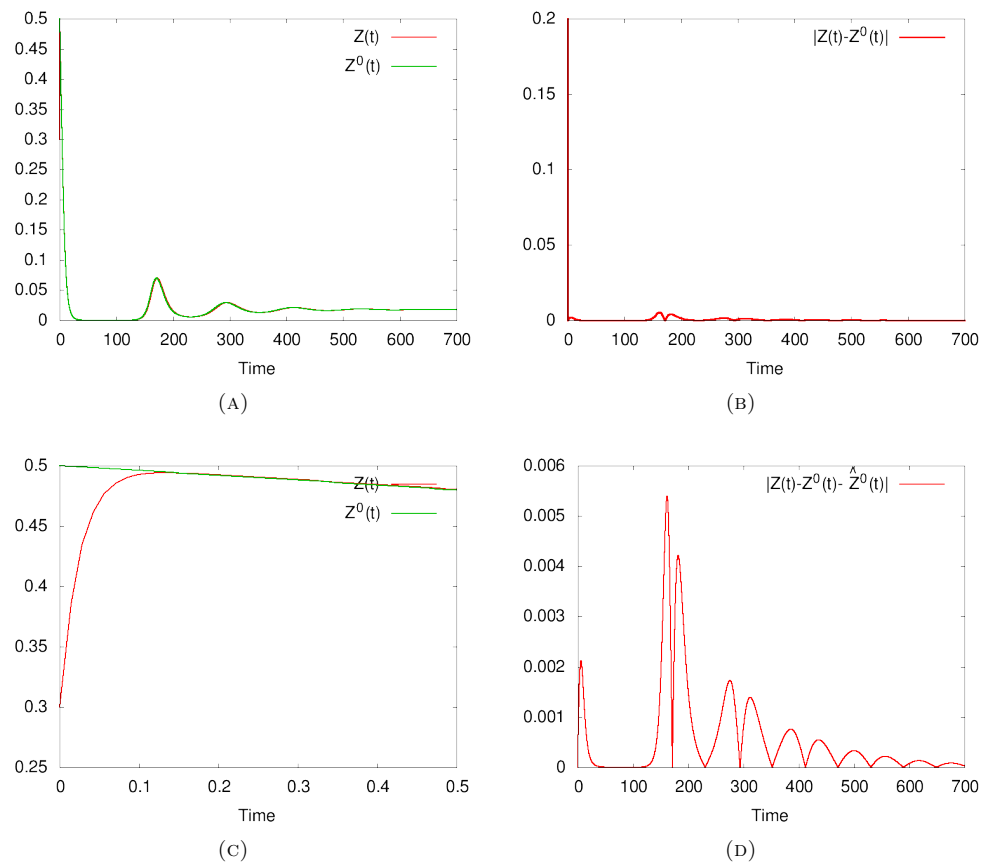


FIGURE 3. Continuation of Figure 5 displaying the infected vector fractions.

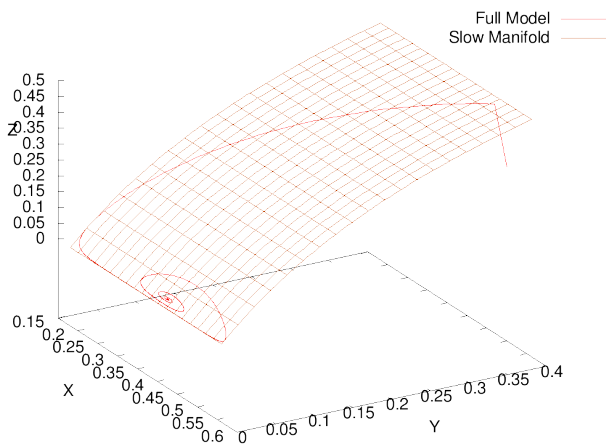


FIGURE 4. Slow manifold dynamics

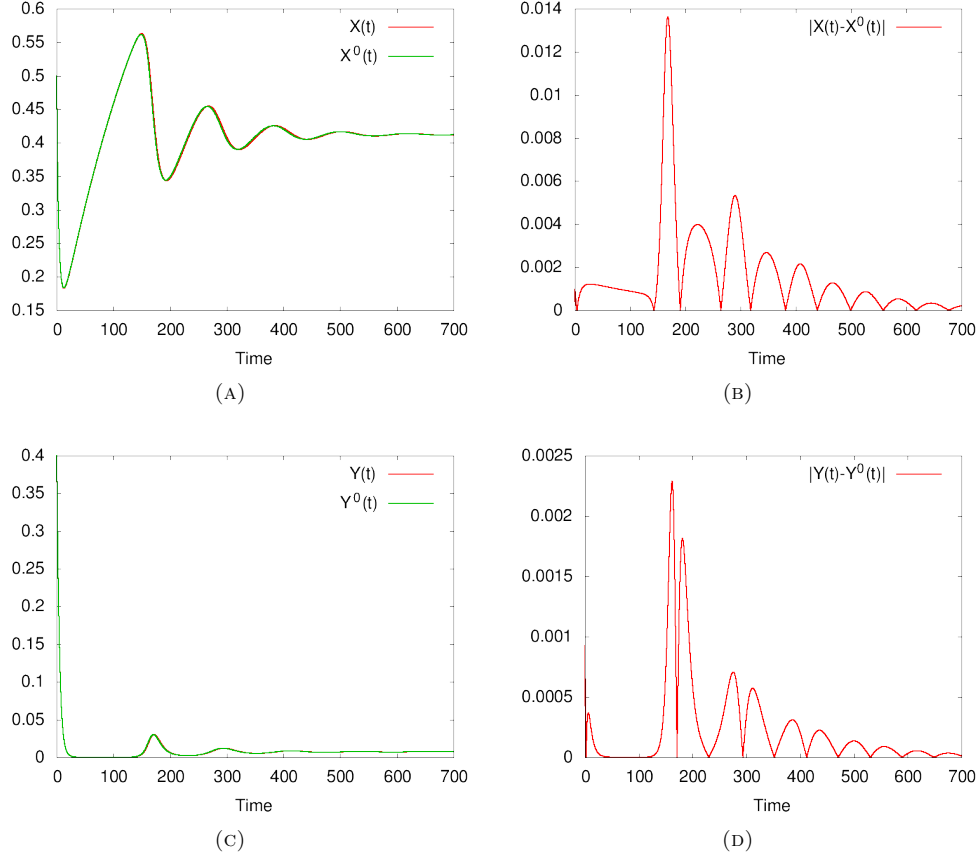


FIGURE 5. Comparison between the full and asymptotic models for the susceptible and infected host fractions. Parameters are the same parameters, except for $\mu_h = 0.005$ and $\gamma = 0.4$. In this case, we have that $R_0 = 2.47$, $D_0 = 0.0125$.

Thus, we are interested in solve

$$(9) \quad \begin{cases} \epsilon \dot{X} &= \bar{\mu}_h(1 - X) - \bar{\delta}XZ \\ \epsilon \dot{Y} &= \bar{\delta}XZ - (\bar{\mu}_h + \bar{\gamma})Y \\ \dot{Z} &= \sigma(1 - Z)Y - \mu_v Z \end{cases}$$

subject to the initial condition

$$X(0) = X_0, \quad Y(0) = Y_0 \quad \text{and} \quad Z(0) = Z_0.$$

In what follows, we formally derive the leading order asymptotic expansion and provide a global analysis together with some numerical results. The proof of asymptottness is very similar to the FVD regime, and hence it is omitted.

A.1. Asymptotic expansion. As before, we let $\epsilon\tau = t$. The asymptotic expansion now take the following form:

$$\begin{aligned} X(t) &= X^0(t) + \hat{X}^0(\tau) + \mathcal{O}(\epsilon), \\ Y(t) &= Y^0(t) + \hat{Y}^0(\tau) + \mathcal{O}(\epsilon), \\ Z(t) &= Z^0(t) + \mathcal{O}(\epsilon). \end{aligned}$$

Here, we shall also have

$$\lim_{\tau \rightarrow \infty} (\hat{X}^0(\tau), \hat{Y}^0(\tau)) = 0.$$

At leading order, we have

$$\begin{aligned} 0 &= \bar{\mu}_h(1 - X^0) - \bar{\delta}X^0Z^0 \\ 0 &= \bar{\delta}X^0Z^0 - (\bar{\mu}_h + \bar{\gamma})Y^0 \\ Z_t^0 &= \sigma(1 - Z^0)Y^0 - \mu_v Z^0 \end{aligned}$$

We can solve for X^0 and Y^0 obtaining

$$X^0 = \frac{\bar{\mu}_h}{\bar{\delta}Z + \bar{\mu}_h} \quad \text{and} \quad Y^0 = \frac{\bar{\delta}\bar{\mu}_h}{\bar{\mu}_h + \bar{\gamma}} \frac{Z^0}{\bar{\delta}Z^0 + \bar{\mu}_h}.$$

Thus the last equation becomes

$$Z_t^0 = \mu_v Z^0 \left(R_0 D_0 \frac{1 - Z^0}{Z^0 + D_0} - 1 \right).$$

Also, we write

$$Z^0(t) = Z^0(\epsilon\tau) = Z^0(0) + \epsilon\tau Z^0(0) + \mathcal{O}(\epsilon^2\tau^2),$$

hence, since $Z^0(0) = Z_0$, we obtain

$$\begin{aligned} \hat{X}_\tau^0 &= \bar{\mu}_h \hat{X}^0 - \bar{\delta} \hat{X}^0 Z_0 \\ \hat{Y}_\tau^0 &= \bar{\delta} \hat{X}^0 Z_0 = (\bar{\gamma} + \bar{\mu}_h) \hat{Y}^0 \end{aligned}$$

The solution can be written as

$$\begin{pmatrix} \hat{X}^0 \\ \hat{Y}^0 \end{pmatrix} = e^{tA} \begin{pmatrix} X_0 - \frac{\bar{\mu}_h}{\bar{\delta}Z_0 + \bar{\mu}_h} \\ Y_0 - \frac{\bar{\delta}\bar{\mu}_h}{\bar{\mu}_h + \bar{\gamma}} \frac{Z_0}{\bar{\delta}Z_0 + \bar{\mu}_h} \end{pmatrix}, \quad A = \begin{pmatrix} -\bar{\mu}_h & -\bar{\delta}Z_0 \\ \bar{\delta}Z_0 & -(\bar{\gamma} + \bar{\mu}_h) \end{pmatrix}.$$

It is straightforward to verify that the eigenvalues of A always have negative real part, and hence that

$$\lim_{\tau \rightarrow \infty} (X^0(\tau), Y^0(\tau)) = 0.$$

A.2. Global stability analysis. The equilibria are $Z_0 = 0$ and $Z_0 = Z^*$. Since

$$\frac{d}{dZ_0} \left(Z_0 \left[R_0 D_0 \frac{1 - Z_0}{D_0 + Z_0} - 1 \right] \right) = R_0 D_0 \frac{1 - Z_0}{D_0 + Z_0} - 1 - R_0 D_0 \frac{1 + D_0}{(D_0 + Z_0)^2} Z_0$$

At $Z_0 = 0$, its value is $R_0 - 1$. So the origin is globally asymptotically stable for $R_0 < 1$.

At $Z_0 = Z^*$ its value is

$$\frac{1 + R_0 D_0}{R_0(1 + D_0)}(1 - R_0)$$

Hence Z^* is globally asymptotically stable, if $R_0 > 1$.

When $R_0 = 1$, we have

$$Z_0 \left[R_0 D_0 \frac{1 - Z_0}{D_0 + Z_0} - 1 \right] = -\frac{D_0 Z_0}{D_0 + Z_0} (Z_0 + 1) < 0, \quad Z_0 \geq 0.$$

Hence, $Z_0 = 0$ is also globally asymptotic stable when $R_0 = 1$.

A.3. Numerical results. The results for the components are qualitatively similar to the fast vector dynamics, and hence are omitted. Nevertheless, the reduction of the dynamics to the slow manifold is more dramatic in this case as shown in figure 6.

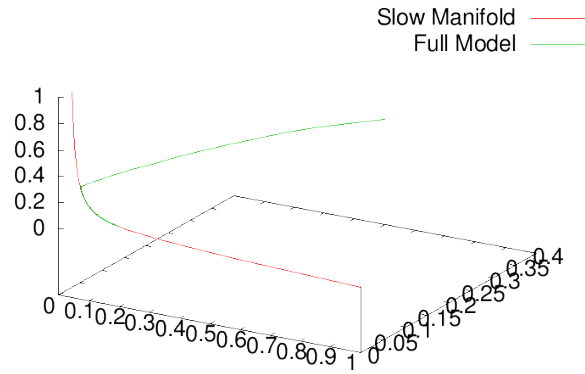


FIGURE 6. Slow manifold dynamics for the fast host regime.

REFERENCES

- [1] N. T. J. Bailey. *The mathematical theory of infectious diseases*. Griffin, 1975.
- [2] L. Cai, S. Guo, X. Li, and M. Ghosh. Global dynamics of a dengue epidemic mathematical model. *Chaos, Solitons & Fractals*, 42(4):2297–2304, Nov. 2009.
- [3] K. Dietz. Transmission and control of arbovirus diseases. In D. Ludwig and C. K. L., editors, *Epidemiology*, pages 104–121. SIAM, 1975.
- [4] L. Esteva and C. Vargas. Analysis of a dengue disease transmission model. *Mathematical Biosciences*, 150(2):131–151, June 1998.
- [5] H. Nishiura. Mathematical and statistical analyses of the spread of dengue. *Dengue Bulletin*, 30:51–67, 2006.
- [6] R. O’Malley. *Singular perturbation methods for ordinary differential equations*. Springer, 1991.
- [7] J. J. Tewa, J. L. Dimi, and S. Bowong. Lyapunov functions for a dengue disease transmission model. *Chaos, Solitons & Fractals*, 39(2):936–941, Jan. 2009.
- [8] H. Yang, H. Wei, and X. Li. Global stability of an epidemic model for vector-borne disease, 2010-04-01.

DEPARTAMENTO DE MATEMÁTICA APLICADA, UNIVERSIDADE FEDERAL FLUMINENSE, R. MÁRIO SANTOS BRAGA, S/N, 22240-920, NITERÓI, RJ, BRASIL.

E-mail address: msouza@mat.uff.br